

## Twisted split category algebras as quasi-hereditary algebras \*

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### Abstract

We show that if  $\mathcal{C}$  is a finite split category,  $k$  is a field of characteristic 0 and  $\alpha$  is a 2-cocycle of  $\mathcal{C}$  with values in  $k^\times$  then the twisted category algebra  $k_\alpha \mathcal{C}$  is quasi-hereditary.

## 1 Introduction

Throughout this paper we assume that  $\mathcal{C}$  is finite category, that is, the objects of  $\mathcal{C}$  form a finite set, and for every  $X, Y \in \text{Ob}(\mathcal{C})$ , the morphism set  $\text{Hom}_{\mathcal{C}}(X, Y)$  is finite. The category  $\mathcal{C}$  is called *split* if, for each morphism  $s \in \text{Hom}_{\mathcal{C}}(X, Y)$ , there is a (not necessarily unique) morphism  $t \in \text{Hom}_{\mathcal{C}}(Y, X)$  such that  $s \circ t \circ s = s$ . Note that  $u := t \circ s \circ t$  then also satisfies  $s \circ u \circ s = s$ , and also  $u \circ s \circ u = u$ . In the special case where  $\mathcal{C}$  has only one object this leads to the notion of a *regular monoid*, see [5].

Let  $k$  be a field, and let  $\alpha$  be a 2-cocycle of  $\mathcal{C}$  with values in  $k^\times$ . That is, for every pair  $s, t \in \text{Mor}(\mathcal{C})$  such that  $t \circ s$  exists, one has an element  $\alpha(t, s) \in k^\times$  such that the following holds: for any  $s, t, u \in \text{Mor}(\mathcal{C})$  such that  $t \circ s$  and  $u \circ t$  exist, one has  $\alpha(u \circ t, s)\alpha(u, t) = \alpha(u, t \circ s)\alpha(t, s)$ . We will study the twisted category algebra  $k_\alpha \mathcal{C}$ , that is, the  $k$ -vector space with basis  $\text{Mor}(\mathcal{C})$  and multiplication

$$t \cdot s := \begin{cases} \alpha(t, s) \cdot t \circ s & \text{if } t \circ s \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

The aim of this paper, see Theorem 3.5, is to show that if  $\mathcal{C}$  is a finite split category and if  $k$  has characteristic 0 then  $k_\alpha \mathcal{C}$  is a quasi-hereditary algebra. This generalizes a result of Putcha, see [9], who proved that regular monoid algebras are quasi-hereditary over  $k = \mathbb{C}$ .

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Our main motivation for studying the quasi-hereditary structure of twisted category algebras comes from the theory of double Burnside rings and biset functors: by a result of Webb, see [10], the category of biset functors over a field of characteristic 0 is a highest weight category. In [1, Example 5.15(b)], we introduced an algebra  $A$  with the property that the category of biset functors (on a finite set of groups) over a field of characteristic 0 is equivalent to the category of  $eAe$ -modules, where  $e$  is an idempotent of  $A$ . Thus, by Webb's result,  $eAe$  is a quasi-hereditary algebra. It is natural to ask whether also  $A$  is quasi-hereditary. In [1] it was also shown that  $A$  is a twisted category algebra for a finite split category. Thus Theorem 3.5 of the present paper, in particular, implies that the algebra  $A$  in [1] is indeed quasi-hereditary.

We further remark that Theorem 3.5 should be of independent interest, since, by work of Wilcox [11], it also covers various prominent classes of cellular algebras (for suitable parameters) such as Brauer algebras, cyclotomic Brauer algebras, Temperley–Lieb algebras, and partition algebras, so that the main result of this paper gives a unified proof for the known fact that these algebras are quasi-hereditary.

We recently learnt that Linckelmann and Stolorz, see [8, Theorem 1.1], independently proved that, under certain conditions on the category, finite twisted category algebras are quasi-hereditary in characteristic 0. These conditions on the category are even weaker than being split, and therefore the results in [8] imply our main result. However, the two approaches are slightly different; for instance, we explicitly determine the radical of the twisted category algebra as part of our proof. We also refer to [8] for a more detailed discussion of the history of proofs that Brauer algebras, Temperley–Lieb algebras, and partition algebras are quasi-hereditary over coefficient fields of characteristic 0.

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## 2 Notation and quoted results

Throughout this section we assume that  $\mathbf{C}$  is finite split category. We begin by collecting some known facts concerning split categories that will be used repeatedly in this paper. For details and proofs of the results quoted here we refer the reader to [7] and [4].

In what follows, given subsets  $S$  and  $T$  of  $\text{Mor}(\mathbf{C})$ , we set  $S \circ T := \{s \circ t \mid s \in S, t \in T \text{ such that } s \circ t \text{ exists}\}$ . In the case where  $S = \{s\}$  or  $T = \{t\}$ , we abbreviate  $S \circ T$  by  $s \circ T$  or  $S \circ t$ , respectively. Note that  $S \circ T$  may be empty, even if neither  $S$  nor  $T$  is empty.

One calls  $S$  a *left ideal* (respectively, *right ideal*) of  $\mathbf{C}$  if  $\text{Mor}(\mathbf{C}) \circ S \subseteq S$  (respectively,  $S \circ \text{Mor}(\mathbf{C}) \subseteq S$ ). Note that this is equivalent to  $\text{Mor}(\mathbf{C}) \circ S = S$  (respectively,  $S \circ \text{Mor}(\mathbf{C}) = S$ ), since every object has an identity morphism. Analogously, one calls  $S$  a (*two-sided*) *ideal* of  $\mathbf{C}$  if  $\text{Mor}(\mathbf{C}) \circ S \circ \text{Mor}(\mathbf{C}) \subseteq S$ , or equivalently, if  $\text{Mor}(\mathbf{C}) \circ S \circ \text{Mor}(\mathbf{C}) = S$ .

**2.1 Idempotents and  $\mathcal{J}$ -classes.** (a) For morphisms  $s, t \in \text{Mor}(\mathbf{C})$  one defines

$$s \mathcal{J} t :\Leftrightarrow \text{Mor}(\mathbf{C}) \circ s \circ \text{Mor}(\mathbf{C}) = \text{Mor}(\mathbf{C}) \circ t \circ \text{Mor}(\mathbf{C}).$$

This yields an equivalence relation  $\mathcal{J}$  on the set  $\text{Mor}(\mathbf{C})$ , and the corresponding equivalence classes are called the  $\mathcal{J}$ -classes of  $\mathbf{C}$ . We will denote the  $\mathcal{J}$ -class of a morphism  $s \in \text{Mor}(\mathbf{C})$  by  $\mathcal{J}(s)$ .

(b) Let  $I$  and  $J$  be  $\mathcal{J}$ -classes of  $\mathbf{C}$ . One sets

$$J \leqslant_{\mathcal{J}} I :\Leftrightarrow \text{Mor}(\mathbf{C}) \circ J \circ \text{Mor}(\mathbf{C}) \subseteq \text{Mor}(\mathbf{C}) \circ I \circ \text{Mor}(\mathbf{C}).$$

Note that this is also equivalent to  $\text{Mor}(\mathbf{C}) \circ s \circ \text{Mor}(\mathbf{C}) \subseteq \text{Mor}(\mathbf{C}) \circ u \circ \text{Mor}(\mathbf{C})$ , where  $s$  and  $u$  are any representatives of  $J$  and  $I$ , respectively. Note further that this defines a poset structure on the set of  $\mathcal{J}$ -classes of  $\mathbf{C}$ .

(c) An *idempotent* of  $\mathbf{C}$  is a pair  $(X, e)$ , where  $X \in \text{Ob}(\mathbf{C})$  and  $e \in \text{End}_{\mathbf{C}}(X)$  is such that  $e \circ e = e$ . For simplicity we will often just write  $e$  instead of  $(X, e)$ .

We say that idempotents  $(X, e)$  and  $(Y, f)$  of  $\mathbf{C}$  are *equivalent* if there exist some  $s \in e \circ \text{Hom}_{\mathbf{C}}(Y, X) \circ f$  and some  $t \in f \circ \text{Hom}_{\mathbf{C}}(X, Y) \circ e$  such that  $e = s \circ t$  and  $f = t \circ s$ . In this case we write  $(X, e) \sim (Y, f)$ , or simply  $e \sim f$ . It is straightforward to show that this defines an equivalence relation on the set of idempotents of  $\mathbf{C}$ ; we will denote the equivalence class of an idempotent  $(X, e)$  by  $[X, e]$  or  $[e]$ .

(d) The next lemma shows that every  $\mathcal{J}$ -class of  $\mathbf{C}$  contains an idempotent. Furthermore, idempotents  $e$  and  $f$  of  $\mathbf{C}$  are equivalent if and only if  $\mathcal{J}(e) = \mathcal{J}(f)$ ; a proof for this can be found in [7, Lemma 2.1]. Thus there is a bijection between the equivalence classes of idempotents of  $\mathbf{C}$  and the  $\mathcal{J}$ -classes of  $\mathbf{C}$ .

**2.2 Lemma** *Let  $s \in \text{Mor}(\mathbf{C})$ , and let  $t, u \in \text{Mor}(\mathbf{C})$  be such that  $s \circ t \circ s = s = s \circ u \circ s$ . Then  $s \circ t$  and  $u \circ s$  are idempotents in  $\mathbf{C}$ . Moreover,*

$$\mathcal{J}(s \circ t) = \mathcal{J}(s) = \mathcal{J}(u \circ s);$$

*in particular,  $s \circ t \sim u \circ s$ .*

**Proof** Clearly,  $s \circ t$  and  $u \circ s$  are idempotents in  $\mathbf{C}$  contained in the  $\mathcal{J}$ -class  $\mathcal{J}(s)$ . Thus, as already mentioned, [7, Lemma 2.1] implies  $s \circ t \sim u \circ s$ .  $\square$

Suppose now that  $k$  is a field, and let  $\alpha$  be a 2-cocycle of  $\mathbf{C}$  with values in  $k^\times$ . The aim of the next section is to prove that, under suitable additional assumptions on  $k$ , the  $k$ -algebra  $k_\alpha \mathbf{C}$  is *quasi-hereditary*. To this end, we summarize here some important facts concerning the algebra  $k_\alpha \mathbf{C}$  and, in particular, its simple modules and its Jacobson radical. For ease of notation, we will henceforth denote the twisted category algebra  $k_\alpha \mathbf{C}$  by  $A$ .

**2.3 Idempotents of  $\mathbf{C}$  and simple  $A$ -modules.** The isomorphism classes of simple  $A$ -modules have been parametrized by Linckelmann–Stolorz [7], generalizing previous work of Ganyushkin–Mazorchuk–Steinberg [4] concerning semigroup algebras.

(a) Given an idempotent  $(X, e)$  of  $\mathbf{C}$ , the group of invertible elements of the monoid  $e \circ \text{End}_{\mathbf{C}}(X) \circ e$  is denoted by  $\Gamma_e$ , and is called a *maximal subgroup* of  $\mathbf{C}$ . Moreover, we set  $J_e :=$

$e \circ \text{End}_{\mathbb{C}}(X) \circ e \searrow \Gamma_e$ . Restricting the 2-cocycle  $\alpha$  to  $\Gamma_e$ , one can view the twisted group algebra  $k_{\alpha}\Gamma_e$  as (non-unitary) subalgebra of  $A$ .

Note also that the element

$$e' := \alpha(e, e)^{-1}e$$

is an idempotent in the algebra  $A$ , and that  $e'Ae' = k_{\alpha}(e \circ \text{End}_{\mathbb{C}}(X) \circ e)$ . Furthermore, there is a  $k$ -vector space decomposition

$$e'Ae' = k_{\alpha}\Gamma_e \oplus kJ_e, \quad (1)$$

$kJ_e$  is a two-sided ideal, and  $k_{\alpha}\Gamma_e$  is a unitary subalgebra of  $e'Ae'$ .

(b) Suppose that  $e$  is an idempotent of  $\mathbb{C}$ , and let again  $e'$  denote the corresponding idempotent in  $A$ . Whenever  $W$  is a  $k_{\alpha}\Gamma_e$ -module, we obtain an  $A$ -module  $Ae' \otimes_{e'Ae'} \tilde{W}$ , where  $\tilde{W}$  is the inflation of  $W$  from  $k_{\alpha}\Gamma_e$  to  $e'Ae'$  with respect to the decomposition (1). In the case where  $W$  is a simple  $k_{\alpha}\Gamma_e$ -module, the  $A$ -module  $Ae' \otimes_{e'Ae'} \tilde{W}$  has a unique simple quotient module; c.f. [6, Section 6.2].

**2.4 Notation** From now on, we denote by  $e_1, \dots, e_n$  representatives of the equivalence classes of idempotents of  $\mathbb{C}$ , and for  $i = 1, \dots, n$  we fix representatives  $T_{i1}, \dots, T_{il_i}$  of the isomorphism classes of simple  $k_{\alpha}\Gamma_{e_i}$ -modules. Moreover, for  $i = 1, \dots, n$  and  $j = 1, \dots, l_i$ , we denote the inflation of the  $k_{\alpha}\Gamma_{e_i}$ -module to  $e'_i Ae'_i$  by  $\tilde{T}_{ij}$ , and the simple head of the  $A$ -module  $Ae'_i \otimes_{e'_i Ae'_i} \tilde{T}_{ij}$  by  $D_{ij}$ . With this, the following holds:

**2.5 Theorem ([7], Theorem 1.2)** *The modules  $D_{ij}$  ( $i = 1, \dots, n, j = 1, \dots, l_i$ ) form a set of representatives of the isomorphism classes of simple  $A$ -modules.*

Denoting the Jacobson radical of  $A$  by  $\mathbf{J}(A)$ , Theorem 2.5 now leads to the following description:

**2.6 Proposition** *With the notation as in 2.4 one has*

$$\mathbf{J}(A) = \{u \in A \mid \forall i = 1, \dots, n : e'_i Au Ae'_i \subseteq kJ_{e_i} + \mathbf{J}(k_{\alpha}\Gamma_{e_i})\}. \quad (2)$$

*In particular, if in addition  $|\Gamma_{e_i}| \in k^{\times}$ , for all  $i = 1, \dots, n$ , then*

$$\mathbf{J}(A) = \{u \in A \mid \forall i = 1, \dots, n : e'_i Au Ae'_i \subseteq kJ_{e_i}\}. \quad (3)$$

**Proof** It suffices to prove that the set on the right-hand side of (2) is the common annihilator of the simple  $A$ -modules  $D_{ij}$  ( $i = 1, \dots, n, j = 1, \dots, l_i$ ). So let  $u \in A$ , let  $i \in \{1, \dots, n\}$ , and let  $j \in \{1, \dots, l_i\}$ . By [1, Lemma 5.6], we know that  $uD_{ij} = 0$  if and only if  $e'_i Au Ae'_i \subseteq$

$\text{Ann}_{e'_i A e'_i}(\tilde{T}_{ij}) = kJ_{e_i} + \text{Ann}_{k_\alpha \Gamma_{e_i}}(T_{ij})$ . Therefore,

$$\begin{aligned} u \in \mathbf{J}(A) &\Leftrightarrow \forall i = 1, \dots, n, j = 1, \dots, l_i: uD_{ij} = 0 \\ &\Leftrightarrow \forall i = 1, \dots, n: e'_i A u A e'_i \subseteq \bigcap_{j=1}^{l_i} (kJ_{e_i} + \text{Ann}_{k_\alpha \Gamma_{e_i}}(T_{ij})) \\ &= kJ_{e_i} + \bigcap_{j=1}^{l_i} \text{Ann}_{k_\alpha \Gamma_{e_i}}(T_{ij}) = kJ_{e_i} + \mathbf{J}(k_\alpha \Gamma_{e_i}), \end{aligned}$$

proving (2). If, moreover  $|\Gamma_{e_i}| \in k^\times$ , for  $i = 1, \dots, n$  then, by [3, Exercise 28.4], the twisted group algebras  $k_\alpha \Gamma_{e_i}$  ( $i = 1, \dots, n$ ) are semisimple, and we derive equation (3).  $\square$

### 3 A heredity chain for $k_\alpha \mathbb{C}$

In this section we will prove the main result, Theorem 3.5. We start by recalling the definition of a quasi-hereditary algebra.

**3.1 Definition (Cline–Parshall–Scott [2])** Let  $k$  be any field. A finite-dimensional  $k$ -algebra  $A$  is called *quasi-hereditary* if there exists a chain

$$\{0\} = J_0 \subset J_1 \subset \dots \subset J_{n-1} \subset J_n = A \quad (4)$$

of two-sided ideals in  $A$  such that, for every  $l = 1, \dots, n$ , when denoting by  $\bar{\cdot}: A \rightarrow A/J_{l-1} = \bar{A}$  the canonical epimorphism, the following conditions are satisfied:

- (i) there is an idempotent  $\bar{e}_l \in \bar{A}$  with  $\bar{J}_l = \bar{A}\bar{e}_l\bar{A}$ ;
- (ii)  $\bar{J}_l \cdot \mathbf{J}(\bar{A}) \cdot \bar{J}_l = \{0\}$ ;
- (iii)  $\bar{J}_l$  is a projective right  $\bar{A}$ -module.

In this case one calls the chain (4) a *heredity chain* for  $A$ .

For the remainder of this section assume again that  $\mathbb{C}$  is a finite split category. For ease of notation we denote from now on the morphism set  $\text{Mor}(\mathbb{C})$  by  $S$ . We will next define a chain of two-sided ideals of  $\mathbb{C}$  that will give rise to a heredity chain for the twisted category algebra in Theorem 3.5.

**3.2 Definition** Let  $e_1, \dots, e_n$  be representatives for the equivalence classes of idempotents of  $\mathbb{C}$ , ordered such that

$$\mathcal{J}(e_i) \leq_{\mathcal{J}} \mathcal{J}(e_j) \quad \text{implies} \quad i \leq j,$$

in which case we also write  $i \leq_{\mathcal{J}} j$ . Moreover, for  $i = 1, \dots, n$ , we define

$$S_i := \mathcal{J}(e_i), \quad S_{\leq_{\mathcal{J}} i} := \bigcup_{j \leq_{\mathcal{J}} i} S_j, \quad S_{\leq i} := \bigcup_{j \leq i} S_j.$$

**3.3 Proposition** *With the notation as in Definition 3.2, for  $i = 1, \dots, n$ , both  $S_{\leq \mathcal{J}i}$  and  $S_{\leq i}$  are ideals of  $\mathbb{C}$ .*

**Proof** Let  $i \in \{1, \dots, n\}$ , let  $s \in S_i$ , and let  $u, v \in S$  be such that  $s \circ u$  and  $v \circ s$  exist. Since  $S \circ s \circ u \circ S \subseteq S \circ s \circ S$  and  $S \circ v \circ s \circ S \subseteq S \circ s \circ S$ , we immediately get  $\mathcal{J}(s \circ u) \leq_{\mathcal{J}} \mathcal{J}(s)$  and  $\mathcal{J}(v \circ s) \leq_{\mathcal{J}} \mathcal{J}(s)$ . Thus  $S_{\leq \mathcal{J}i}$  is an ideal of  $\mathbb{C}$ , for  $i = 1, \dots, n$ , and since  $S_{\leq i} = \bigcup_{j \leq i} S_{\leq \mathcal{J}j}$ , the latter is an ideal of  $\mathbb{C}$  as well.  $\square$

**3.4 Proposition** (a) *Let  $s, t \in S$  be such that  $s = s \circ t \circ s$ . Then  $s \circ S = s \circ t \circ S$ .*

(b) *Let  $l \in \{1, \dots, n\}$ , and let  $s, t \in S_l$  be such that  $s \circ S \subseteq t \circ S$ . Then  $s \circ S = t \circ S$ .*

(c) *Let  $l \in \{1, \dots, n\}$ , and let  $s, t \in S_l$ . Then the sets  $(s \circ S)_l := (s \circ S) \cap S_l$  and  $(t \circ S)_l := (t \circ S) \cap S_l$  are either equal or disjoint.*

(d) *Let  $l \in \{1, \dots, n\}$ . There is a subset  $\epsilon$  of  $[e_l]$  such that the sets  $(e \circ S)_l := (e \circ S) \cap S_l$  ( $e \in \epsilon$ ) form a partition of  $S_l$ .*

**Proof** (a) This follows from  $s \circ S = s \circ t \circ s \circ S \subseteq s \circ t \circ S \subseteq s \circ S$ .

(b) Let  $q, r \in S$  be such that  $s \circ q \circ s = s$  and  $t \circ r \circ t = t$ , and set  $e := s \circ q$  and  $f := t \circ r$ . Since  $s \in t \circ S$ , the idempotents  $e$  and  $f$  are endomorphisms of the same object of  $\mathbb{C}$ , say  $X$ . By Part (a), we have  $e \circ S \subseteq f \circ S$  and it suffices to show that  $f \circ S \subseteq e \circ S$ . Recall that  $\mathcal{J}(s) = S_l = \mathcal{J}(t)$ . Thus, by Lemma 2.2, we have  $e \sim f$ , so that there exist  $u \in e \circ S \circ f$  and  $v \in f \circ S \circ e$  with  $e = u \circ v$  and  $f = v \circ u$ . Since  $e \circ S \subseteq f \circ S$  we also have  $f \circ e = e$ . Note that  $u$  and  $v$  are endomorphisms of  $X$ . Since  $\text{End}_{\mathbb{C}}(X)$  is finite, there exist positive integers  $a$  and  $b$  such that  $u^{b+a} = u^b$ . Composition with  $v^b$  from the left yields  $u^a = f$ . In fact, we have  $v \circ u = f$  and  $f \circ u = f \circ e \circ u = e \circ u = u$ . Finally, we obtain  $e \circ f = e \circ u^a = u^a = f$  which implies that  $f \in e \circ S$  and  $f \circ S \subseteq e \circ S$ .

(c) Assume that  $(s \circ S)_l \cap (t \circ S)_l$  is non-empty and that  $u \in (s \circ S)_l \cap (t \circ S)_l$ . Then we obtain  $u \circ S \subseteq s \circ S$  and  $u \circ S \subseteq t \circ S$ . Now Part (b) yields  $s \circ S = u \circ S = t \circ S$  and  $(s \circ S)_l = (t \circ S)_l$ .

(d) Clearly,  $S_l$  is the union of its subsets  $(s \circ S)_l$ ,  $s \in S_l$ , and by Part (a) also of the subsets  $(e \circ S)_l$ ,  $e \in [e_l]$ ,  $l = 1, \dots, n$ . The condition  $(e \circ S)_l = (f \circ S)_l$  defines an equivalence relation on the set  $[e_l]$ . If  $\epsilon$  is a set of representatives for the corresponding equivalence classes then, by Part (c),  $S_l$  is the disjoint union of the subsets  $(e \circ S)_l$ ,  $e \in \epsilon$ .  $\square$

**3.5 Theorem** *Let  $\mathbb{C}$  be a finite split category and let  $\alpha$  be a 2-cocycle of  $\mathbb{C}$  with values in the multiplicative group  $k^\times$  of a field  $k$ . Assume further that, for each idempotent  $e$  of  $\mathbb{C}$ , the order of  $\Gamma_e$  is invertible in  $k$ . With the notation as in Definition 3.2, let  $J_i := kS_{\leq i}$ , for  $i = 1, \dots, n$ , and let  $J_0 := \{0\}$ . Then*

$$\{0\} = J_0 \subset J_1 \subset \dots \subset J_n = k_\alpha \mathbb{C} \quad (5)$$

*is a heredity chain for  $k_\alpha \mathbb{C}$ . In particular, the twisted category algebra  $k_\alpha \mathbb{C}$  is quasi-hereditary.*

**Proof** We set  $A := k_\alpha \mathbf{C}$ . Since  $S_{\leq i}$  is an ideal of  $S$ ,  $J_i$  is an ideal of  $A$  for all  $i = 0, \dots, n$ . We show that the chain (5) satisfies conditions (i)–(iii) in Definition 3.1. For this, let  $l \in \{1, \dots, n\}$ , and again let  $\bar{\cdot}: A \rightarrow A/J_{l-1}$  denote the canonical epimorphism.

By definition, we have  $S \circ s \circ S = S \circ e_l \circ S$ , for every  $s \in S_l$ , thus  $S_l \subseteq S \circ e_l \circ S$ . From this we get  $\bar{J}_l = \bar{A} \bar{e}_l \bar{A}$ , and we have verified condition (i).

Next we verify condition (ii). Note that, since  $A$  is a finite-dimensional algebra over a field, we have  $\overline{\mathbf{J}(A)} = \mathbf{J}(\bar{A})$ . Hence it suffices to show that  $sut \in J_{l-1}$ , for all  $s, t \in S_{\leq l}$  and all  $u \in \mathbf{J}(A)$ . If  $s \in S_{\leq l-1}$  or  $t \in S_{\leq l-1}$  then this is clearly true. So we may suppose that  $s, t \in S_l$ . Let  $q, r \in S$  be such that  $s \circ q \circ s = s$  and  $t \circ r \circ t = t$ . Then  $S \circ s \circ S = S \circ e_l \circ S = S \circ t \circ S$  and  $[e_l] = [s \circ q] = [q \circ s] = [t \circ r] = [r \circ t]$ , by Lemma 2.2. So there exist elements  $x \in q \circ s \circ S \circ e_l$ ,  $y \in e_l \circ S \circ q \circ s$ ,  $v \in t \circ r \circ S \circ e_l$ , and  $w \in e_l \circ S \circ t \circ r$  such that

$$q \circ s = x \circ y = x \circ e_l \circ y, \quad t \circ r = v \circ w = v \circ e_l \circ w, \quad e_l = y \circ x = w \circ v.$$

Since  $u \in \mathbf{J}(A)$ , Proposition 2.6 implies  $(e_l \circ y)u(v \circ e_l) \in e'_l A u A e'_l \subseteq kJ_{e_l}$ . Furthermore, we have  $e_l \circ S \circ e_l \subseteq S_{\leq l}$ , since  $e_l \in S_l \subseteq S_{\leq l}$  and since, by Proposition 3.3,  $S_{\leq l}$  is an ideal in  $S$ . By [7, Lemma 2.6], it is known that  $(e_l \circ S \circ e_l) \cap S_l = \Gamma_{e_l}$ , hence  $J_{e_l} = (e_l \circ S \circ e_l) \setminus \Gamma_{e_l} \subseteq S_{\leq l-1}$ , thus  $kJ_{e_l} \subseteq J_{l-1}$ . This implies  $(e_l \circ y)u(v \circ e_l) \in J_{l-1}$  and we obtain

$$sut = (s \circ q \circ s)u(t \circ r \circ t) = (s \circ x \circ e_l \circ y)u(v \circ e_l \circ w \circ t) \in J_{l-1},$$

as required.

It remains to verify condition (iii). By Proposition 3.4(d), we know that  $\bar{J}_l$  is the direct sum of right ideals of the form  $\bar{e} \bar{A}$ , for suitable idempotents  $e \in A$ . Since each such summand is a projective right  $\bar{A}$ -module, so is  $\bar{J}_l$ , and the proof of (iii) is complete.  $\square$

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